

Thermodynamics of a one-dimensional system of point bosons: comparison of the traditional approach with a new one

Maksim Tomchenko

Bogolyubov Institute for Theoretical Physics,
14b, Metrolohichna Str., Kyiv 03680, Ukraine
E-mail:mtomchenko@bitp.kiev.ua

Abstract

We compare two approaches to the construction of the thermodynamics of a one-dimensional periodic system of spinless point bosons: the Yang–Yang approach and a new approach proposed by the author. In the latter, the elementary excitations are introduced so that there is only one type of excitations (as opposed to Lieb’s approach with two types of excitations: particle-like and hole-like). At the weak coupling, these are the excitations of the Bogolyubov type. The equations for the thermodynamic quantities in these approaches are different, but their solutions coincide (this is shown below and is the main result). Moreover, the new approach is simpler. An important point is that the thermodynamic formulae in the new approach for any values of parameters are formulae for an ensemble of quasiparticles with the Bose statistics, whereas a formulae in the traditional Yang–Yang approach have the Fermi-like one-particle form.

1 Introduction

The modern physics of one-dimensional (1D) many-particle systems is divided, in fact, into two areas: physics of point particles and physics of nonpoint ones. Moreover, these areas are not fairly joined, and not all connections between them are clear. Lieb’s approach [1] involves particle-like and hole-like excitations. The dispersion law of the particle-like excitations coincides with that for a system of nonpoint bosons [2, 3, 4, 5] (though, this is verified only for a weak coupling and periodic boundary conditions). But none of the models of a system of nonpoint bosons find the hole-like excitations (see reviews [6, 7]). This is the first mismatch. On the basis of the Lieb’s approach, Yang and Yang constructed the thermodynamics of a system of point bosons [8], where the formulae have the Fermi-like form, though the thermodynamics of a system of nonpoint bosons is described by the formulae for an ensemble of Bose quasiparticles [9]. This is the second mismatch. These two mismatches of the theories of point and nonpoint particles are usually referred to (1) particular properties of a point interaction in the 1D case and, sometimes, (2) the drawbacks of the models for nonpoint particles. Note that some models of a system of nonpoint particles [2, 5] use a condensate. We recall that a uniform 1D system of nonpoint bosons can possess a quasicondensate close to the true condensate, if the system is finite, the coupling is very weak, and the temperature is low [10].

For the fermions, the hole-like excitations are a characteristic property in the cases of point and nonpoint interactions. The history and the analysis of models of a 1D system of point fermions can be found in [11, 12, 13, 14].

A new way of introduction of elementary excitations for a system of point bosons is proposed in the recent work [15]. In this approach, there is only *one* type of excitations and the thermodynamic formulae coincide with those for nonpoint bosons. However, no comparison of the thermodynamics with that in the traditional Yang–Yang approach [8]

was made. Below, we will carry out such comparison and will see that the new [15] and traditional [8] approaches are equivalent. However, the new approach is simpler and removes both above-mentioned mismatches of the theories of point and nonpoint bosons. We note that the thermodynamics of 1D systems of point bosons was developed [16, 17, 18, 19], but these methods are based on the Yang–Yang approach [8]. However, the approach [15] is different in essence.

2 Basic equations and ways of introduction of quasiparticles

In the present work, we will compare the traditional Yang–Yang approach [8] to the thermodynamics of spinless point bosons with the new one [15]. The Yang–Yang approach was analyzed in [12] in detail. These approaches differ by different ways of introduction of elementary excitations. Therefore, we consider firstly these ways. The system of N point bosons is described by the Schrödinger equation [20]

$$-\sum_j \frac{\partial^2}{\partial x_j^2} \Psi + 2c \sum_{i < j} \delta(x_i - x_j) \Psi = E \Psi, \quad i, j = 1, \dots, N. \quad (1)$$

Here, we take $c > 0$ and use the units with $\hbar = 2m = k_B = 1$. The theory of penetrable point bosons started by the classical works by Lieb and Liniger [20] and by Lieb [1]. In work [20], the following equations for quasimomenta k_i of a periodic system of N bosons were found:

$$(-1)^{N-1} e^{-ik_j L} = \exp \left(-2i \sum_{s=1}^N \arctan \frac{k_s - k_j}{c} \right), \quad j = 1, \dots, N, \quad (2)$$

where L is the size of the system. The analysis of work [20] is based on the equation for the quantity $k_{j+1} - k_j$. It was shown [8] that Eqs. (2) yield the equations

$$Lk_j = 2\pi I_j - 2 \sum_{l=1}^N \arctan \frac{k_j - k_l}{c}, \quad j = 1, \dots, N. \quad (3)$$

The ground state of the system corresponds to the quantum numbers $I_j \equiv I_j^{(0)} = j - \frac{N+1}{2}$ [11, 21]. For such quantum numbers, Eqs. (3) are similar to the equations for the “Fermi sea,” and the elementary excitations can be divided into hole-like and particle-like [1]. However, Gaudin noticed [11, 21] that, with the help of the equality $\arctan \alpha = (\pi/2) \operatorname{sgn}(\alpha) - \arctan(1/\alpha)$, Eqs. (3) can be written in the form

$$Lk_j = 2\pi n_j + 2 \sum_{l=1}^N \arctan \frac{c}{k_j - k_l} \Big|_{l \neq j}, \quad j = 1, \dots, N, \quad (4)$$

where n_j are integers, and the equality $I_j = n_j + j - \frac{N+1}{2}$ holds. Equations (4) follow from (3) at the ordering $k_1 < k_2 < \dots < k_N$, which requires that the inequalities $I_1 < I_2 < \dots < I_N$ and $n_1 \leq n_2 \leq \dots \leq n_N$ be satisfied. For the ground state, all quantum numbers n_j are the same: $n_j = 0$. Therefore, all N equations (4) are equivalent. This corresponds to the Bose symmetry. On the basis of Eqs. (4), a quasiparticle is defined very simply [15]: The elementary excitation (quasiparticle) is associated with a change in one of n_j by any integer (as compared with the ground state, for which $n_j = 0$ for all j). A change in n_j with the other j means the creation of one more quasiparticle, and so on. In this case, index j enumerates quasiparticles, and the value of n_j characterizes the

j -th quasiparticle, by determining its momentum and energy (see [15]). For such way of introduction of quasiparticles, it is possible to calculate the statistical sum for the system with $N = \infty, L = \infty$. As a result, the same formula for the total free energy, as for the one-dimensional He II [9], is obtained [15]:

$$F = E_0 + T \sum_{l \neq 0} \ln \left(1 - e^{-E(l)/T} \right). \quad (5)$$

Here, E_0 is the ground-state energy of the system, and $E(l)$ means the energy of a quasiparticle (for the free bosons, the same formula [9, 22] is true, where $E_0 = 0$ and $E(l)$ is the energy level of a boson). For the periodic boundary conditions (BCs), l runs the same values as any n_j in system (4): $l = \pm 1, \pm 2, \dots$ ($l = 0$ does not enter (5), because zero n_j corresponds to the absence of an excitation). For the zero BCs, formula (5) is valid as well, but l runs the values $1, 2, 3, \dots$ [15]. In the limit $N, L \rightarrow \infty, N/L = \text{const}$, the energy levels of the system with periodic and zero BCs coincide [15]. It follows from (5) that the values of thermodynamic parameters for such systems also coincide. The assertions made for zero BCs are valid, if the equations under zero BCs (they are similar to (4)) have the unique solution; the uniqueness was proved in [23]. Let us call the method [15] the n -approach, in view of n_j in Eqs. (4).

We note that the procedure in [15] cannot be repeated on the basis of Eqs. (3). This is related to the following. The ground state corresponds to the collection of successive integer (or half-integer) numbers $I_j^{(0)}$. To obtain an elementary excitation, one of the numbers $I_j^{(0)}$ should be changed by an integer. But if the changed I_l will coincide with one of the remaining $I_j^{(0)}$, we obtain the solution with two identical k_j , which is forbidden [20]. In order that the changed I_l will not coincide with one of the remaining $I_j^{(0)}$, we should take out I_l outside the bounds of the “Fermi sea” of the numbers $I_j^{(0)}$. This leads to hole-like and particle-like excitations of the Lieb’s picture. However, the transition to the system (4) changes the situation radically, because the coincidence of several n_j is admissible for (4): This does not lead to the coincidence of k_j , which allows us to introduce quasiparticles with the Bose statistics. Thus, the key point in the definition of quasiparticles is the structure of the equations for k_j .

It is of importance to understand the interconnection between different approaches: particle-like and hole-like excitations by Lieb, “holes” and k ’s by Yang and Yang [8], and quasiparticles in the n -approach. In the literature, k ’s by Yang and Yang are frequently called “particles.” It is probably not quite suitable term, since such “particle” is sometimes confused with Lieb’s particle-like excitation. The latter is the excitation of the whole system, which arises at the transfer of I_j from the “Fermi sea” surface outward (in this case, I_N increases by a natural number s , or I_1 decreases by s). The hole-like excitation by Lieb corresponds to the transfer of I_j from the depth of the Fermi sea to the surface (this is equivalent to the increase in *several* numbers $I_{N-l}, I_{N-l+1}, \dots, I_N$ by 1). The hole- and particle-like excitations are similar, respectively, to a “hole” inside the Fermi sea and to a hole on the Fermi sea surface. Yang and Yang [8] referred a hole to a change in any I_j and did not use the term “particle.” If $I_l^{(0)}$ from the collection $\{I_j^{(0)}\}$ changes, then all k_j in (3) change, and k_l changes above all. Therefore, the input value of k_l seemed to come out from the distribution of $\{k_j\}$. Yang and Yang associated a “hole” with such k_l . The remaining k_j vary slightly and are called k ’s [8]. It is worth noting that, in the Yang–Yang approach, a hole and each k_j from k ’s is a single number, whereas the excitations by Lieb are collective, since they include changes in all numbers k_j .

Consider two simple examples. Equations (4) with $n_j = 0$ for all j describe the ground

state. Let us increase n_N by 1. According to the n -approach, this creates a quasiparticle. In view of the relation $I_j = n_j + j - \frac{N+1}{2}$, our action increases I_N by 1. Within Lieb's approach, this means the creation of a particle-like excitation. In the Yang–Yang approach, the “hole” k_N is created and the remaining k 's are slightly shifted. We now pass from the last state (with $n_{j \leq N-1} = 0$, $n_N = 1$) to the state with $n_{j \leq N-2} = 0$, $n_{N-1} = n_N = 1$. In the n -approach language, this means the creation of the second quasiparticle. In the Yang–Yang approach, the second hole (k_{N-1}) appears, and k 's (k_1, \dots, k_{N-2}) are shifted. Within Lieb's approach, a particle-like excitation disappears and a hole-like excitation, corresponding to the shift of I_{N-1} , I_N by 1, is created. A hole-like excitation corresponds in the n -approach to several quasiparticles with the same (minimal) momentum. Lieb's language is not very suitable due to a complicated connection of quasiparticles with quantum numbers I_j and to the separation of excitations into particle-like and hole-like ones. The Yang–Yang approach uses also two quantities (holes and k 's), and the visual picture is not quite simple, but such language is efficient for the construction of the thermodynamics. The n -approach is simpler than the approach by Lieb (due to a single type of quasiparticles and a simpler connection of quasiparticles with quantum numbers n_j) and than the Yang–Yang approach (since a quasiparticle is considered as a single object, whereas the approach Yang–Yang considers separately all k_j instead of a single quasiparticle). However, these three approaches are mathematically equivalent.

Let us pass to the formulae. Here, we consider a system with periodic BCs, because the thermodynamics in [8] was constructed for a periodic system. First, let us consider the n -approach. It is convenient to pass from formula (5) to the equivalent formula [15]

$$F = E_0 + \frac{TL}{2\pi} \int_{-\infty}^{\infty} dp \ln \left(1 - e^{\frac{-E(|p|)}{T}} \right), \quad (6)$$

where $E(|p|)$ and p are, respectively, the energy and the momentum of a quasiparticle. For low T , it is the exact formula for the free energy of a 1D system of point bosons. As is seen, the formula is written in the language of quasiparticles. The analogous formula is known for a system of nonpoint bosons [9, 22]. In [22], this formula was deduced from properties of the ensemble of quasiparticles. In the same way, we can obtain formula (6) for a system of point bosons, if the quasiparticles are defined in the n -approach (Lieb's quasiparticles are not characterized by Bose symmetry; therefore, formula (6) is wrong for them). We note that formula (6) was obtained in [15] in a different way, by the direct summation of the partition function.

The thermodynamic relation [22]

$$dF = -SdT - PdV + \mu dN \quad (7)$$

yields the formula for the total entropy

$$S = -\frac{\partial F}{\partial T} \Big|_{N, V = \text{const}}. \quad (8)$$

Formula (6) and the dispersion law $E(|p|)$ completely assign the thermodynamics of the system. Importantly, that formula (6) for all values of T and n ($n = N/L$) is a formula for the gas of noninteracting quasiparticles with Bose statistics. In this case, the approach [15] involves only one type of quasiparticles. For all $\gamma = c/n$, their dispersion law coincides with the dispersion law of particle-like excitations by Lieb.

Yang and Yang have obtained the following thermodynamic formulae [8]:

$$F = N\mu - PL, \quad P = \frac{T}{2\pi} \int_{-\infty}^{\infty} dk \ln \left(1 + e^{\frac{-\epsilon(k)}{T}} \right), \quad (9)$$

$$\epsilon(k) = -\mu + k^2 - \frac{Tc}{\pi} \int_{-\infty}^{\infty} dq \frac{\ln\left(1 + e^{-\frac{\epsilon(q)}{T}}\right)}{c^2 + (k - q)^2}, \quad (10)$$

$$dP = (S/L)dT + (N/L)d\mu, \quad (11)$$

$$n \equiv \frac{N}{L} = \frac{\partial P}{\partial \mu}|_{T=\text{const}}, \quad (12)$$

$$S = L \frac{\partial P}{\partial T}|_{\mu=\text{const}}. \quad (13)$$

These formulae have the Fermi-like form and are deduced on the basis of the ideology of the Fermi sea of numbers $I_j^{(0)}$.

To calculate the thermodynamic quantities within the n -approach [15], it is necessary to find $E(p)$ from a linear integral equation (see Eqs. (2.18)–(2.20) in [1] or (43), (44), (55) in [15]) and then to determine the free energy F from (6). In the Yang–Yang approach, the problem is more complicated: We need to determine $\epsilon(k)$ as a function of c, T, μ from the nonlinear integral equation (10); to find $\mu(n, T, c)$ from (12); and then to determine F and the entropy S from (9), (13).

We note that formulae (6) and (9)–(13) are true for $N, L \rightarrow \infty$, $N/L = \text{const}$. In this case, formulae (9)–(13) are valid for any T , but formula (6) holds only at low T , for which the interaction of quasiparticles can be neglected [15].

3 Calculation of thermodynamic quantities in two approaches

Thus, we have two systems of equations: (6)–(8) and (9)–(13). We will calculate from them the values of F , S and will compare the results. We consider only the regimes of weak and infinitely strong couplings, for which the formulae for $E(|p|)$ are available and the solutions for F and S can be found in the n -approach (6)–(8) easily.

1. Regime of weak coupling: $\gamma = c/n \ll 1$. For the n -approach, we need to know the dispersion law of quasiparticles [15] and the ground-state energy [20]. They are given by the Bogolyubov's formulae for point particles:

$$E(p) = \sqrt{p^4 + 4cnp^2}, \quad E_0 = Ncn(1 - 4\sqrt{\gamma}/(3\pi)). \quad (14)$$

Let us substitute $E(p)$ in (6). After the integration by parts and a transformations, we get

$$F = E_0 - \frac{\pi T^2 L}{6v_s} I_f(a), \quad a = \frac{v_s^2}{T}, \quad v_s = 2\sqrt{cn}, \quad (15)$$

$$I_f(a) = \frac{6}{\pi^2} \int_0^\infty \frac{dx}{\sqrt{1 + x^2/a^2}} \frac{x + 2x^3/a^2}{e^{x\sqrt{1+x^2/a^2}} - 1}. \quad (16)$$

Here and below, v_s is the velocity of sound. Analogously, formulae (8), (6), and (14) yield

$$S = \frac{\pi TL}{3v_s} I_s(a), \quad I_s(a) = \frac{6}{\pi^2} \int_0^\infty \frac{dx}{\sqrt{1 + x^2/a^2}} \frac{x + 3x^3/(2a^2)}{e^{x\sqrt{1+x^2/a^2}} - 1}. \quad (17)$$

The values of $I_f(a)$ and $I_s(a)$ are presented in Fig. 1. For $a \gg 1$, we have $I_f \approx I_s \approx 1$.

Formulae (6)–(8) hold if the number of quasiparticles N_{qp} is much less than the number of particles N [15]. The criterion is as follows: $N_{qp} \lesssim 0.1N$. The numerical solution of Eqs.

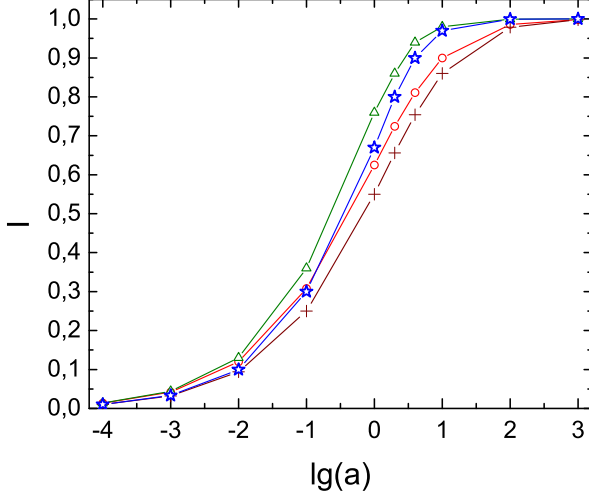


Figure 1: [Color online] Functions $I_f(a)$ ($\triangle\triangle\triangle$), $I_s(a)$ ($\star\star\star$), $I_f^\infty(a)$ ($\circ\circ\circ$), and $I_s^\infty(a)$ ($+++$), see Eqs. (16), (17), (27), and (28), correspondingly. In this and other figures, $\lg(x) \equiv \log_{10} x$.

(4) indicates that, for $N_{qp} \gtrsim 0.1N$, the interaction of quasiparticles changes noticeably their energy. Therefore, formulae (5) and (6) become not quite true. We now find N_{qp} . With regard for (5), the total internal energy of the system is

$$U = F + TS = F - T \frac{\partial F}{\partial T} \Big|_{N,V=const} = E_0 + \sum_{l \neq 0} \frac{E(l)}{e^{E(l)/T} - 1}. \quad (18)$$

On the other hand, since

$$U = E_0 + \sum_{l \neq 0} E(l) \bar{N}_l, \quad (19)$$

we can find the mean number of Bose quasiparticles with energy $E(l)$:

$$\bar{N}_l = (e^{E(l)/T} - 1)^{-1}. \quad (20)$$

From whence, the total number of quasiparticles is

$$N_{qp} = \sum_{l \neq 0} \bar{N}_l = \sum_{p=-\infty}^{\infty} (e^{E(p)/T} - 1)^{-1} \Big|_{p \neq 0} = 2 \sum_{j=1}^{\infty} \left(e^{\frac{v_s p_1 j \sqrt{1+j^2 p_1^2 / v_s^2}}{T}} - 1 \right)^{-1} \approx \frac{2}{q_1} \ln Q^{-1}, \quad (21)$$

where $p = 2\pi l/L$, $Q = q_1$ for $q_1 \geq q_2$ and $Q = q_2$ for $q_1 < q_2$, $q_1 = v_s p_1/T$, $q_2 = p_1/v_s$, $p_1 = 2\pi/L$; it is assumed that $N, L \rightarrow \infty, N/L = const$. The condition $N_{qp} \ll N$ requires

$$\frac{T}{2n^2 \sqrt{\gamma}} = \xi \ll \frac{\pi}{\ln Q^{-1}}. \quad (22)$$

For large L and N , condition (22) can be written as $\frac{T}{2n^2 \sqrt{\gamma}} \ll \frac{\pi}{\ln(\sqrt{\gamma}N)}$. Condition (22) indicates that, for large L and N , the temperature T should be low. Relation (22) implies that the value of T decreases, if L increases. For $L = \infty$, we have $T = 0$. This is absurd result. Why did such paradox appear? Formula (20) follows from (5), and formula (5) is derived from the canonical Gibbs distribution. This distribution is obtained usually [9, 22] from Liouville's theorem for an ensemble of identical closed equilibrium systems and the division of each such system into a small subsystem and the much larger thermostat. But if

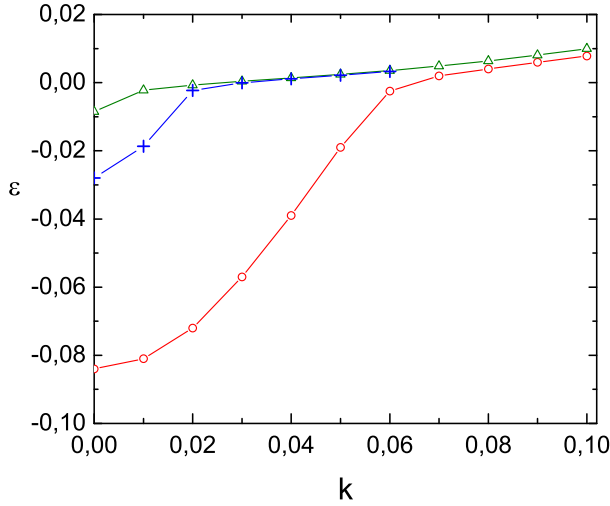


Figure 2: [Color online] Function $\epsilon(k)$ as a solution of the Yang–Yang equations (24) and (12) for $c = 0$ ($\triangle\triangle\triangle$), $c = 0.0001$ ($+++$), and $c = 0.001$ ($\circ\circ\circ$). For $k > 0.05$, the curves approach one another and the asymptote $\epsilon = k^2 - \mu$. All points are determined for $T = 0.001$ and $n = 1$. Each curve corresponds to the own value of μ , which is determined from Eq. (12). For free bosons, $\epsilon(k) = T \ln(e^{\frac{k^2 - \mu}{T}} - 1)$ [8] and $\mu = -2.064 \cdot 10^{-7}$. For $c = 0.0001$ and 0.001 , we found $\mu \approx 2c$.

our system is infinite and unbounded, it cannot be a part of the “much larger” system (even the imaginary one) of the same dimension. Therefore, the Gibbs distribution is applicable only for a finite system. In addition, the equilibrium arises due to the interaction of parts of the system. Therefore, the equilibrium in a system with infinite N and V is established in infinite time. That is, the equilibrium in an infinite system makes no sense. In the classical book by Gibbs [24], the canonical distribution was obtained from the condition of equilibrium in the system $[\partial\rho(q_1, \dots, q_N, p_1, \dots, p_N, t)/\partial t = 0]$ and Liouville’s theorem for an ensemble of systems ($d\rho/dt = 0$). Gibbs noted that only the consideration of systems with a finite partition function has meaning [24]. However, the infinite systems with infinite number of degrees of freedom and with a realistic interparticle interaction are characterized, presumably, by the infinite partition function. In our opinion, the above paradox arose due to the application of the Gibbs distribution to an infinite system. We may apply the Gibbs distribution to a finite system and then pass to the infinite system in formulae (to simplify calculations, e.g.). By returning then to the finite system, we expect to get a reasonable results. However, in this case, we may obtain unphysical results for some properties at $N, V = \infty$. In work [15], we used the canonical Gibbs distribution for finite N and L ; then, we set $N = \infty$, $L = \infty$ and obtained formula (5). But we assumed that this formula holds also for finite N and L .

In the traditional approach, it is necessary to solve the integral equation (10). By making in this equation changes $k \rightarrow -k$ and (under the sign of integral) $q \rightarrow -q$, we get the same equation for the function $\epsilon(-k)$. This means that

$$\epsilon(-k) = \epsilon(k). \quad (23)$$

Therefore, instead of (10), we can solve the equation

$$\epsilon(k) = -\mu + k^2 - \frac{Tc}{\pi} \int_0^\infty dq \ln \left(1 + e^{\frac{-\epsilon(q)}{T}} \right) \left(\frac{1}{c^2 + (k - q)^2} + \frac{1}{c^2 + (k + q)^2} \right), \quad (24)$$

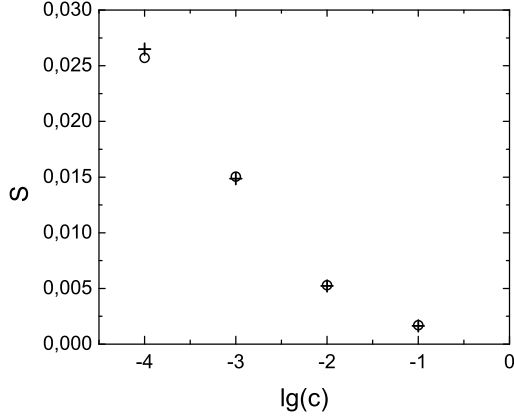


Figure 3: Function $S(c)$ derived for the regime of weak coupling in the traditional ($\circ \circ \circ$) and new ($+++$) approaches. For all points, we have $n = 1$ and $T = 0.001$. The traditional approach corresponds to Eqs. (23), (24), (12), and (13), and the new one corresponds to formulae (17).

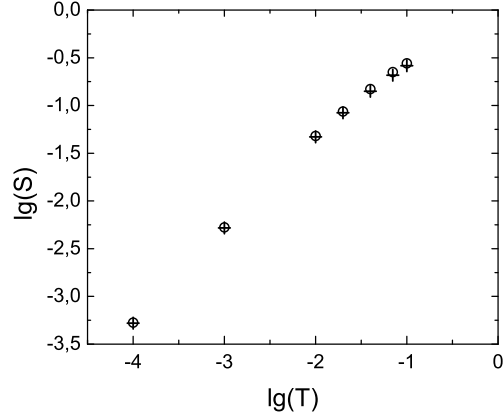


Figure 4: Function $S(T)$ obtained for the regime of weak coupling ($n = 1$, $c = 0.01$) in the traditional ($\circ \circ \circ$) and new ($+++$) approaches.

where $k \geq 0$. We tried several numerical procedures, but only the method of iterations worked (one needs usually about 500 iterations). The solutions for $\epsilon(k)$ at $c \ll 1$ are shown in Fig. 2.

Let us find the chemical potential μ for the known concentration n . We need to find $\epsilon(k)$ for μ and $\mu + \delta\mu$ with a small $\delta\mu$ and to substitute those $\epsilon(k)$ in Eqs. (9) and (12): $n = \frac{P(\mu+\delta\mu)-P(\mu)}{\delta\mu}|_{T=const.}$. Analogously, we obtain the entropy from (9) and (13): $S = L \frac{P(T+\delta T)-P(T)}{\delta T}|_{\mu=const.}$. The free energy can be found from (9). Since formula (15) is valid for low temperatures, the term $\frac{\pi T^2 L}{6v_s} I_f$ in (15) is usually small as compared with E_0 . However, the entropy S is determined namely by this small term. Therefore, we now calculate the entropy, which allow us to verify formula (15) to within a small correction $\sim T^2$.

In Figs. 3 and 4, we show the solutions for $S(c)$ and $S(T)$ found in the traditional and new approaches. As is seen, both approaches give the *identical* solutions (a difference of 1–2% is connected with errors of the numerical method). For almost all points in Figs. 3 and 4, we have $\xi \ll 1$, so that criterion (22) is satisfied (if L is large, but not too large; ξ is not small ($\xi \lesssim 0.5$) near the point $T = 0.1$ in Fig. 4). We have $\mu \approx 2c$ for all points in Fig. 3 and $\mu \approx 0.02$ for all points in Fig. 4. We also derived the solutions $F(n)$ and $S(n)$ for fixed $c = T = 0.001$. In this case, for $n = 0.5$ – 2 , we have $\mu \approx n/500$, and the solutions for S in the traditional and new approaches coincide. The solutions for F coincide as well.

In the n -approach, μ is zero, because the quasiparticles with the Bose statistics can be freely created and annihilated. All nonzero μ are related to the Yang–Yang approach, where the system is described in the one-particle language.

For $c = 0.0001$ – 0.1 , we derived the solution $\mu \approx 2c$ at $n = 1$, $T = 0.001$. This is of interest, because this requires $\mu \approx 0$ for $c = 0$. However, Eqs. (9)–(13) yield for $c = 0$ the equations for free bosons (they are given in [8]). In this case, μ is a solution of Eq. (12) and is *negative* for all n and T . For example, for $n = 1$ and $T = 0.001$ (parameters of Fig. 3), we get $\mu \approx -2.064 \cdot 10^{-7} < 0$. These results show that, most likely, (1) $\mu > 0$ for all $c > 0$ and (2) at $c = 0$, the value of μ decreases *by jump* to some $\mu < 0$ (hence, F and S

vary also by jump at $c = 0$). Though a smooth transition from a positive $\mu(c = 0.0001)$ to a negative $\mu(c = 0)$, as c decreases from 0.0001 down to 0, is also possible. Fig. 2 shows the solutions for $\epsilon(k)$ at $c = 0; 0.0001, 0.001$; these solutions admit both possibilities. At both smooth and jump-like transition $\mu(c > 0.0001) \rightarrow \mu(c = 0)$, the value of μ turns to zero at some $c = c_{cr} \geq 0$. But we failed to find c_{cr} . To clarify whether a jump exists, it is necessary to find a solution of the Yang–Yang equations on the set of points of the domain $0 < c < 0.0001$.

In the new approach, the situation is as follows. For $\gamma \ll 1$, $c \rightarrow 0$, and $n, T = \text{const}$, we have $a = 4cn/T \rightarrow 0$. The numerical analysis indicates that, at $a \rightarrow 0$, $I_f \approx 1.4\sqrt{a}$, $I_s \approx 1.05\sqrt{a}$. Therefore, formulae (15)–(17) yield $F \approx E_0 - 1.4\pi LT^{3/2}/6$, $S \approx 1.05\pi LT^{1/2}/3$. Since $E_0(c = 0) = 0$, we have $F \approx -2TS/3$. For free bosons, $F = -TS$ (this relation can be obtained from formulae in [8]). The difference of the factors $2/3$ and 1 is related to the jump of F and (or) S or to the fact that, at $c \rightarrow 0$, condition (22) is violated, and formulae (15)–(17) become not quite proper.

The jump of F and (or) S at $c = 0$ is possible due to the transition to the thermodynamic limit. Indeed, for a finite system, the minimal $|p|$ in the Bogolyubov formula $E(p) = \sqrt{p^4 + 4cnp^2}$ is equal to $|p| = 2\pi/L$. If we pass to the thermodynamic limit for arbitrarily small c , then L can be taken so large that, at smallest $|p|$, the relation $p^4 \ll 4cnp^2$ will hold. Therefore, the dispersion law will be linear in p , which yields formulae (15)–(17). At fixed finite L , the value of c could be taken so small that the dispersion law would be quadratic in p at small p , like $E(p)$ for free particles. Therefore, the thermodynamic solutions for small c would undoubtedly pass continuously to the solutions for $c = 0$. However, for an infinite system, the point $c = 0$ is singular for the dispersion law and, therefore, can be singular for the thermodynamic quantities as well. We recall that the above-discussed paradoxical conclusion that the condition $N_{qp} \ll N$ requires $T = 0$ is also related to the transition to the thermodynamic limit.

It is worth noting that, for a 1D system of point bosons, an analog of the phase transition at the point $\mu = 0$ was found [18, 19] for the regime $\gamma \gg 1$. As far as we see, the analyticity of the thermodynamic functions in μ and T is conserved in a vicinity of the point $\mu = 0$, which corresponds to the proof [8]. But it is unclear whether this peculiarity reveals in the n -approach (for which μ can be only zero, because the physics is determined by Bose quasiparticles) and, if yes, what is the physical meaning of this peculiarity?

In Fig. 5, we present the curves $F(T)$ found in the new and traditional approaches. These curves practically coincide. In particular, the curve $F(T)$ of the Yang–Yang method approaches at $T \rightarrow 0$ the asymptote $F = E_0$ corresponding to formula (15) of the n -approach. With the increase in T , the results of the traditional and new solutions become somewhat different. This is related to the fact that ξ becomes large (e.g., for $T = 0.1$, we have $\xi \approx 0.5$), and therefore, condition (22) is broken. For the points with $T \lesssim 0.02$ we have $\xi \lesssim 0.1$.

2. Regime of infinitely strong coupling: $c = +\infty$, $\gamma = +\infty$. Under the description in the language of atoms, it is the Fermi-like regime [8] (though the wave function has the Bose symmetry). But, under the description in the language of n -quasiparticles [15], we have the purely bosonic regime.

First, we consider the n -approach [15]. The dispersion law of quasiparticles and the ground-state energy are determined by the Girardeau’s formulae [25]:

$$E^\infty(p) = p^2 + 2\pi n|p|, \quad E_0^\infty = Nn^2\pi^2/3. \quad (25)$$

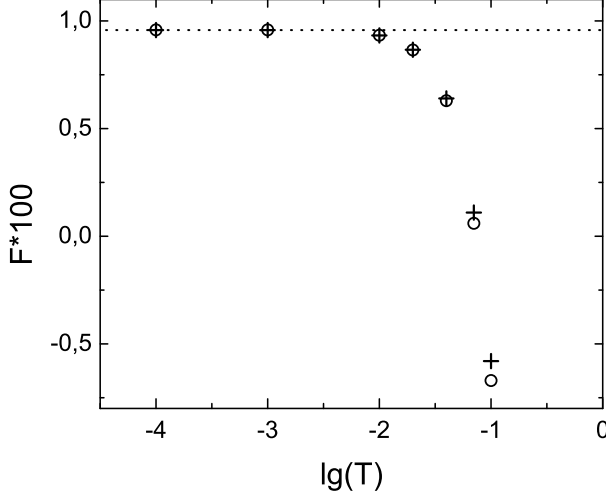


Figure 5: Function $F(T)$ derived for the regime of weak coupling ($n = 1$, $c = 0.01$) in the traditional (○ ○ ○) and new (+ + +) approaches. The traditional approach corresponds to Eqs. (23), (24), (12), (9), and the new one — to formulae (15), (16). The value of F is increased by 100 times. The dotted line shows the asymptote $F(T \rightarrow 0) = E_0 = 0.00958$.

Substitute $E^\infty(p)$ in (6) and make some transformations, then we obtain:

$$F = E_0^\infty - \frac{\pi T^2 L}{6 v_s^\infty} I_f^\infty(a^\infty), \quad a^\infty = \frac{(v_s^\infty)^2}{T}, \quad v_s^\infty = 2\pi n, \quad (26)$$

$$I_f^\infty(a^\infty) = \frac{6}{\pi^2} \int_0^\infty dx \frac{x + 2x^2/a^\infty}{e^{x(1+x/a^\infty)} - 1}. \quad (27)$$

Formulae (8), (6), and (25) yield

$$S = \frac{\pi T L}{3 v_s^\infty} I_s^\infty(a^\infty), \quad I_s^\infty(a^\infty) = \frac{6}{\pi^2} \int_0^\infty dx \frac{x + 3x^2/(2a^\infty)}{e^{x(1+x/a^\infty)} - 1}. \quad (28)$$

Formulae for F and S are *the same* as those for the regime of weak coupling. The difference consists only in the changes $E_0 \rightarrow E_0^\infty$, $v_s \rightarrow v_s^\infty$, $I_f(a) \rightarrow I_f^\infty(a^\infty)$, $I_s(a) \rightarrow I_s^\infty(a^\infty)$. The values of $I_f^\infty(a^\infty)$ and $I_s^\infty(a^\infty)$ are shown in Fig. 1. We have $I_f^\infty \approx I_s^\infty \approx 1$ for $a^\infty \gg 1$ and $I_f^\infty \approx 1.4\sqrt{a^\infty}$, $I_s^\infty \approx 1.04\sqrt{a^\infty}$ for $a^\infty \ll 1$.

Formulae (26), (28) with $I_s^\infty = I_f^\infty = 1$ were obtained previously [17, 18] for the regime $T \rightarrow 0$, $\gamma \rightarrow \infty$ in a more complicated way from the Yang–Yang equations [8] (in works [17, 18], there is the small slip in the formula for v_c (our v_s^∞): It should be $v_s^\infty = (\frac{1}{m} \frac{\partial P}{\partial n})^{1/2} = \frac{\hbar n \pi}{m} = 2\pi n$, then the formula for $F(T)$ in [17, 18] coincides with (26) with $I_f^\infty = 1$). Still before, the formula for F , close to (26), was obtained by the field-theoretic method [26].

Let us find out the consequences of the condition $N_{qp} \ll N$. At $N, L \rightarrow \infty$, we obtain

$$N_{qp} = \sum_{p=-\infty}^{\infty} (e^{E^\infty(p)/T} - 1)^{-1} \Big|_{p \neq 0} = 2 \sum_{j=1}^{\infty} \left(e^{q_1 j (1+q_2 j)} - 1 \right)^{-1} \approx \frac{2}{q_1} \ln Q^{-1}, \quad (29)$$

where $Q = \max(q_1, q_2)$, $q_1 = v_s^\infty p_1 / T$, $q_2 = p_1 / v_s^\infty$, $p_1 = 2\pi / L$. Relation $N_{qp} \ll N$ yields

$$T \ll \frac{(v_s^\infty)^2}{2 \ln Q^{-1}}. \quad (30)$$

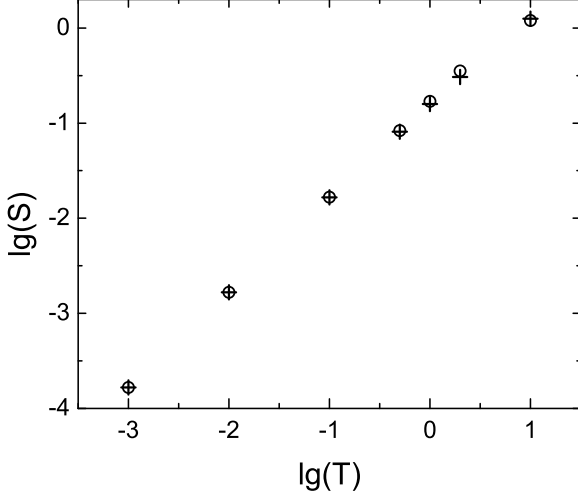


Figure 6: Function $S(T)$ for the regime of strong coupling ($c = +\infty$, $n = 1$) obtained in the traditional ($\circ \circ \circ$) and new ($+ + +$) approaches. The traditional approach corresponds to Eqs. (32) and (33), and the new one — to formulae (28).

For not too large L , we have $\ln Q^{-1} \sim 10$, and (30) is approximately reduced to $T \ll n^2$.

Let us turn to the Yang–Yang approach. For $c = \infty$, Eq. (10) has the solution $\epsilon(k) = -\mu + k^2$. Equations (9)–(13) yield the formulae

$$P = \frac{2T\sqrt{T}}{\pi} \int_0^\infty dq \frac{q^2}{e^{q^2-\eta} + 1}, \quad (31)$$

$$n = \frac{\partial P}{\partial \mu}|_{T=\text{const}} = \frac{\sqrt{T}}{\pi} \int_0^\infty \frac{dq}{e^{q^2-\eta} + 1}, \quad (32)$$

$$S = L \frac{\partial P}{\partial T}|_{\mu=\text{const}} = \frac{N\sqrt{T}}{\pi n} \int_0^\infty dq \frac{3q^2 - \eta}{e^{q^2-\eta} + 1}, \quad (33)$$

where $\eta = \mu/T$. In this case, $S = \frac{3LP}{2T} - \frac{N\mu}{T}$. Therefore, we have $PL = \frac{2ST}{3} + \frac{2N\mu}{3}$, which yields

$$F = N\mu - PL = \frac{NT}{3} \left(\eta - \frac{2S}{N} \right). \quad (34)$$

Let the concentration n be known. Then, we numerically derive η from Eq. (32) and S , F from (33), (34). For $T \rightarrow 0$ and $\mu > 0$ in (32), we have $e^{q^2-\eta} \rightarrow 0$ for all $q^2 < \eta$. Therefore, $\eta(T \rightarrow 0) = \pi^2 n^2 / T + \varphi(T/n^2)$. The numerical analysis indicates that $\varphi(\frac{T}{n^2} \rightarrow 0) \approx 0.1 \frac{T}{n^2}$. At $T \rightarrow 0$ and $\mu \leq 0$, Eq. (32) yields $n \rightarrow 0$.

In Fig. 6, we show the solution $S(T)$ found numerically from Eqs. (32), (33) as compared with solution (28) in the n -approach. It is seen that both solutions coincide with good accuracy for $T < 1$ and are slightly different for $T \gtrsim 1$. The last is because the ratio N_{qp}/N becomes large (of the order of 1 for $N = 10^4, n = 1$) for $T \gtrsim 1$. Therefore, the approximation of free quasiparticles [15] becomes improper. For $T \leq 0.01$, the difference of solutions (28) and (33) for $S(T)$ is at most 0.1%. Solutions for $F(T)$, (26) and (34), coincide for small T with even higher accuracy (we do not portray them).

For the regime of strong coupling, the entropy of the quasi-1D Bose gas was measured [27] and approximately agrees with the solution of the Yang–Yang equations.

For small T , the thermodynamics is defined by the sound part of the dispersion law. Therefore, formulae (15) and (17) (with $I_s = I_f = 1$ and the value of v_s corresponding to γ under consideration) should be valid for *any* coupling constant γ .

A 1D system of fermions with the holon and spinon excitations was considered in [28]. For small T and any coupling constant, the following formula was obtained [28]:

$$F = E_0 - \frac{\pi T^2 L}{6} \left(\frac{1}{v_s^{(1)}} + \frac{1}{v_s^{(2)}} \right). \quad (35)$$

It is a natural generalization of formula (15) to the case of a system with two types of quasiparticles (the holon dispersion curve is gapped, the spinon one is gapless; in both cases, $v_s^{(i)} = \frac{\partial E^{(i)}}{\partial k}|_{E^{(i)} \rightarrow 0}$). Therefore, we assume that the n -approach is also applicable to other integrable systems.

Thus, for $\gamma \ll 1$ and $\gamma = +\infty$, the solutions in the new approach [15] coincide with the corresponding solutions in the Yang–Yang approach [8]. We have no doubts that such coincidence holds also for the intermediate values of γ .

4 Discussion of the results and experiments

The thermodynamics of a 1D system of point bosons can be constructed within the traditional Yang–Yang method and the new n -method [15]. These approaches are equivalent. In the previous section, we have seen that they lead to the same results for F and S . However, the new approach seems to be somewhat more physical, in the following sense. At small temperatures, any excited state of a gas is most simply described as a set of quasiparticles. The n -approach is constructed namely in the language of quasiparticles and, therefore, leads to simpler equations. In this case, the quasiparticles are characterized by the same statistics, as the quasiparticles in a system of nonpoint bosons. As for the Yang–Yang approach, it applies the language of individual atoms, and the formulae do not correspond to a definite statistics, generally speaking. However, the Yang–Yang approach is more universal, since it allows one to find the thermodynamic quantities at any temperature (the n -approach works only at small T).

In the recent years, some interesting experiments with a quasi-1D Bose gas in a trap were carried out [29, 30]. The application of the Bragg spectroscopy allowed one to get the detailed experimental data on a dynamical structural factor. In particular, those data give information about the dispersion law for quasiparticles. The authors of works [29, 30] made conclusion about the essential contribution of the hole-like and particle-like excitations to the scattering, because at $\gamma > 3$ the scattering peak is placed *between* the energy of a particle-like excitation and the energy of a hole-like excitation [29]. According to the conclusion [30], the inhomogeneity of the system affects slightly the peak width, the broadening has a non-temperature nature and is related to the interaction (of separate atoms, apparently).

Our impression from the results [29, 30] is the following. The vibrations or rotations of the cloud as a whole have an insignificant influence on the Bragg spectroscopy; the inhomogeneity of a gas affects slightly the peak width [30]. Therefore, the physics of the system should be defined by quasiparticles, like for a uniform quantum liquid, such as He II. That is, the broadening of the experimental peak is related, in our opinion, to the usual temperature mechanism (interaction of quasiparticles). It is interesting that the experimental peak deviates from the energy of particle-like quasiparticles (see Fig.

2 in [29]), if γ increases. We recall that the n -approach is equivalent to the Lieb's and Yang–Yang approaches. Moreover, a Lieb's quasiparticles (of both types) can be presented as one or several n -quasiparticles. The energy of Lieb's particle-like excitation coincides with the energy of n -quasiparticle [15] (at the same momentum, of course). But the n -approach involves only n -quasiparticles. Therefore, it is strange that the experimental peak deviates from the energy of this quasiparticle. This can be related to not quite accurate determination of some parameters of the system or to the too simplified description of the system. The more radical possibility consists in that the solutions for point bosons do not coincide with the solutions for real nonpoint bosons. However, the observation of only one peak [29] agrees with the n -picture, because the last contains one type of excitations. If the system would have two independent types of excitations, then we would observe two peaks.

The picture with “holes” and “particles” took deep root, but our approach is simpler and presumably more physical. Therefore, it is worth attempting to interpret the experimental data in the language of this approach.

It is also noteworthy that a hole-like excitation, which presents several co-directed phonons (according to the n -approach), is related to solitons [31, 32, 33, 34].

The future problem is the construction of the thermodynamics for a 1D system of finite size. For the infinite system, the thermodynamic quantities depend on T analytically [8]. Does this analyticity conserve also for a finite system? For a 1D system of point bosons, the number of quasiparticles does not exceed the number of atoms. Therefore, for a finite system, we must take $\eta_l = 0, 1, 2, \dots, N$ and $\sum_l \eta_l \leq N$ in the partition function (see Eq. (65) in [15]). Such sum is easily calculated only for $N = \infty$ [15].

5 Summary

Since the first works [25, 20, 1, 8] till now, the one-dimensional system of spinless point bosons is described in the language of fermions. The fermionicity is manifested in the properties of Eqs. (3), presence of hole-like excitations, and Fermi-like structure of the thermodynamic equations [8]. In the present work and in [15], we have shown that the point bosons can be described in the purely bosonic language, by using Eqs. (4) instead of equivalent ones (3). In our approach, we have only one type of quasiparticles. At a weak coupling, they are Bogolyubov quasiparticles. One succeeded in constructing the thermodynamics for the infinite system in the language of quasiparticles [15]. In this case, the method is essentially different from the Yang–Yang method [8] and give the ordinary formulae for an ensemble of noninteracting Bose quasiparticles, like for a system of nonpoint bosons (e.g., for He II). It is not too strange, because the point bosons are the limiting case of nonpoint ones. In the present article, we have shown that the solutions for the thermodynamic quantities in the new [15] and traditional [8] approaches *coincide*. Thus, a 1D system of spinless point bosons can be described in both bosonic and fermionic languages. This is of interest, since only the bosonic language is developed for nonpoint bosons. Apparently, the approach [15] can be applied also to other integrable systems. In particular, some Fermi systems without a pairing can probably be described in a bosonic language.

Moreover, we have found the evidence of a possible jump of the thermodynamic quantities of infinite system at the coupling constant $c = 0$.

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